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# SIMPLE PDE MODEL OF SPOT REPLICATION IN ANY DIMENSION*

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**Abstract.** We propose a simple PDE model which exhibits self-replication of spot solutions in any dimension. This model is analyzed in one and higher dimensions. In the radially symmetric case, we rigorously demonstrate that a weakened version of the conditions proposed by Nishiura and Ueyama for self-replication are satisfied. In dimension three, two different types of replication mechanisms are analyzed. The first type is due to radially symmetric instability, whereby a spot bifurcates into a ring. The second type is nonradial instability, which causes a spot to deform into a peanut-like shape and eventually split into two spots. Both types of replication are observed in our model, depending on parameter choice. Numerical simulations are shown confirming our analytical results.

**Key words.** reaction-diffusion systems, PDEs, self-replication

**AMS subject classifications.** 35K57, 35Q92

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## 1. Introduction.

In this paper we present a simple nonautonomous PDE which exhibits the self-replication of a spot solution in $\mathbb{R}^N$, $N \geq 1$. The PDE is

$$u_t = \Delta u - u + \frac{(1 + a|x|^q)\, u^p}{\int_{\mathbb{R}^N} (1 + a|x|^q) u^{p+1} dx}, \quad x \in \mathbb{R}^N; \qquad \nabla u(0,t) = 0, \tag{1}$$

Examples of this phenomenon are shown in Figure 1. Self-replication was first observed by Pearson in the Gray–Scott model [23]. Since then, many theoretical and numerical studies have looked at self-replication in both one and two spatial dimensions for the Gray–Scott model in different parameter regimes [25], [24], [21], [22], [19], [4], [3], [14], [5]. Many other reaction-diffusion systems have been found to exhibit self-replication behavior. These include the ferrocyanide-iodide-sulfite system [11], the Belousov–Zhabotinsky reaction [12], [18], the Gierer–Meinhardt model [16], [10], [15], the Bonhoffer–van-der-Pol type system [7], [8], [9], and the Brusselator [13].

In an effort to classify reaction-diffusion systems that can exhibit pulse self-replication, Nishiura and Ueyama, motivated by the numerical study of the Gray–Scott model, proposed a set of necessary conditions for this phenomenon to occur in [21]. Roughly stated, these conditions are the following:

(S1) The disappearance of the ground-state solution due to a fold point (saddle-node bifurcation) that occurs when a control parameter is increased above a certain threshold value.

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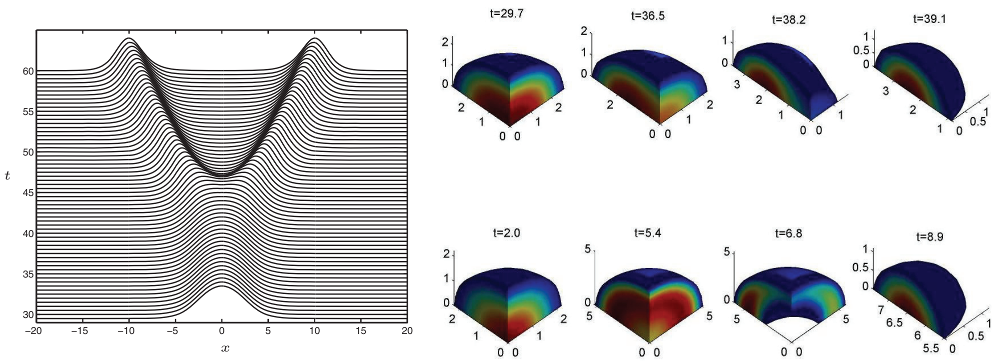



(a) (b)

Fig. 1. (a) *Numerical simulation of* (1) *in one dimension with* $p = 2, q = 2, a = 0.08$. *Self-replication is observed.* (b) *Numerical simulation of* (1) *in three dimensions, showing two different types of self-replication. The snapshots show the cross section of the solution in the first quadrant* $x, y, z > 0$. *The surface corresponds to the contour* $u = 0.6 \max(u)$; *cross sections* $x = 0$ *and* $y = 0$ *are shown in a color map (online) with red corresponding to* $\max(u)$ *and blue to* $0.6 \max(u)$. First row: *Spot-to-spot bifurcation due to instability of a nonradial eigenfunction. The parameters are* $p = 2, q = 1.3$, *and* $a = 0.5$. Second row: *spot-to-ring bifurcation due to radial instability. The parameters are* $p = 2, q = 3, a = 0.035$. *The spot-to-ring bifurcation is followed by ring-to-spot bifurcation.*

(S2) The existence of a dimple eigenfunction at the fold point, which is believed to be responsible for the initiation of the self-replication process. By definition, a dimple eigenfunction is a radially symmetric eigenfunction $\Phi(|x|)$ associated with a zero eigenvalue at the fold point that decays as $|x| \to \infty$ and that has a positive zero (see Figure 3).

(S3) Stability of the steady state solution on one side of the fold point.

(S4) The alignment of the fold points, so that the disappearance of $K$ ground states, with $K = 1, 2, 3, \ldots$, occurs at roughly the same value of the control parameter.

These conditions are believed to be necessary (although not sufficient) for an initiation of the self-replication event. They were first verified numerically for a certain regime of the Gray–Scott model in [21], [6]. In a different regime, the Gray–Scott model reduces to the so-called core problem [19], [5], [14]. After a scaling, the core problem is

$$\left\{ \begin{array}{c} U_{rr} + \frac{N-1}{r} U_r - U + U^2 V = 0; \quad V_{rr} + \frac{N-1}{r} V_r - a U^2 V = 0; \\ V(0) = 1; \quad V'(0) = 0 = U'(0); \\ V, U > 0; \quad U \to 0 \text{ as } r \to \infty. \end{array} \right. \tag{2}$$

The existence of a fold point of (2) (condition (S1)) in one dimension was demonstrated numerically in [19]. Conditions (S2) and (S3) were also numerically verified for (2) in [14]. More recently, the following weaker version of condition (S1) was shown analytically in [5]:

(S1*) The steady state ceases to exist if a control parameter is increased above a certain threshold value.

There are few analytical results for (2) in two or three dimensions (but see [19] for some partial results).

In this paper we show analytically that the simple model (1) can exhibit self-replication in any dimension for some parameter values of $p, q$ as $a$ is sufficiently increased from zero. We analytically verify condition (S1$^*$) under the following assumptions:

$$\begin{gathered} p>1 \text{ and } \ q>\frac{(p-1)N}{2} \text{ if } N=1 \text{ or } 2, \\ 1<p<\frac{N+2}{N-2} \text{ and } q>\frac{(p-1)\,(N-1)}{2} \text{ if } N\geq 3. \end{gathered} \tag{3}$$

Provided that these assumptions hold, conditions (S2) and (S3) also hold under an additional hypothesis that (S1) holds. In this case, a single self-replication event is observed numerically in (1) as the parameter $a$ is increased past some critical value $a_c$. In one dimension, the bifurcation structure and the self-replication mechanism are analogous to what has been observed for the reduced Gray–Scott model (2); however, unlike the studies [19], [5], we are able to verify not only condition (S1$^*$) but also conditions (S2) and (S3) analytically, under an additional hypothesis that (S1) holds.

It appears difficult to verify condition (S1) analytically, even for this simplified model: only the weaker condition (S1$^*$) is rigorously shown to hold under assumptions (3). Based on numerical evidence, we conjecture that (S1) holds under the same assumptions.

In dimensions two and three, the self-replication conditions (S1)–(S3) lead to a radially symmetric bifurcation, whereby a spot bifurcates into a ring that concentrates on the surface of an $N$-dimensional ball. However, there is another self-replication mechanism that can occur. Namely, a spot can become unstable with respect to nonradial perturbations of mode 2. Numerically, this leads to what we shall call *peanut splitting*, whereby a radially symmetric spot starts to acquire a peanut-like shape, which eventually pinches off and becomes two spots. We study both types of self-replication of (1) in three dimensions; we demonstrate that both are possible depending on choice of parameters (see Figure 1(b)). Analytically, we show that when $N=3,\ p=2$, and $q=1$, the spot will undergo peanut splitting if $a$ is sufficiently large, whereas no spot-to-ring bifurcation is expected for any value of $a$. On the other hand, if $p=2,\ q>1$, both radial and nonradial splitting is possible. For $q$ sufficiently large, the radial splitting dominates as illustrated in Figure 1(b), row 2. To the best of our knowledge, this is the first rigorous demonstration of self-replication in three dimensions.

The summary of the paper is as follows. In section 2 we study the steady state problem associated with (1). The main result is Theorem 2, which proves the boundedness of the bifurcation diagram under assumptions (3), thus verifying the condition (S1$^*$). In section 3.1 we study radial stability and analytically verify conditions (S2) and (S3). This fully characterizes self-replication in one dimension and also characterizes radial replication in dimensions $>1$. In section 3.2 we address nonradial instability to complete the classification of self-replication phenomena in three dimensions. In section 4 we discuss some generalizations, compare our model to other models with self-replication, and provide some open problems and concluding remarks.

**2. Analysis of the ground state.** We start our analysis by considering the radially symmetric positive ground state solution of (1) which satisfies

$$u_{rr}+\frac{N-1}{r}u_r-u+c_1u^p\left(1+ar^q\right)=0,\qquad u'(0)=0,\quad u\to 0 \text{ as } r\to\infty,\quad u>0,$$

where

$$c_1 := \frac{1}{c_0 \int_0^\infty r^{N-1}(1 + ar^q)u^{p+1}dr}$$

and $c_0$ is the surface area of a sphere in $\mathbb{R}^N$. Next we scale $u = c_1^{1/(1-p)}\tilde{u}$. After dropping the tilde, the ground state solution satisfies

$$u_{rr} + \frac{N-1}{r}u_r - u + u^p (1 + ar^q) = 0, \quad u'(0) = 0, \quad u \to 0 \text{ as } r \to \infty. \tag{4}$$

It is well known that the steady state problem (4) with $a = 0$ admits a unique solution when $p \in (1, p^*)$, where

$$p^* = \begin{cases} \frac{(N+2)}{(N-2)}, & N \geq 3, \\ \infty, & N \leq 2, \end{cases} \tag{5}$$

is the critical exponent [2], [17]. Starting from $a = 0$ we wish to examine how the solution depends on $a$. For a fixed $a$, define $s := u(0)$ and let $s_0 := s(0)$ be the height of the solution with $a = 0$. To show that the solution also exists with $a > 0$, consider the linearized problem at $a = 0, s = s_0$:

$$\Delta\phi - \phi + pu^{p-1}\phi = \lambda\phi.$$

The kernel of the operator $\phi \to \Delta\phi - \phi + pu^{p-1}\phi$ is spanned by $\{u_{x_1}, \dots u_{x_n}\}$; this operator is invertible when restricted to radially symmetric functions. (See [27] for more details.) It follows that there exists a solution to (4) whenever $a$ is sufficiently close to zero, with $s$ close to $s_0$, using a bifurcation argument similar to the one of Crandall and Rabinowitz [1]. The detailed proof of this local existence is given in Appendix C. We summarize the result as follows.

LEMMA 1. *Suppose that $p \in (1, p^*)$. For all sufficiently small $a$, there exists a $C^1$ function $s(a)$ and a solution $u(r; a)$ to* (4) *with the following properties:* (i) $s(a) = u(0; a)$; (ii) $u(r; a) > 0$; (iii) $s(0) = s_0$; *and* (iv) $u(r, a)$ *is $C^1$ in $a$.*

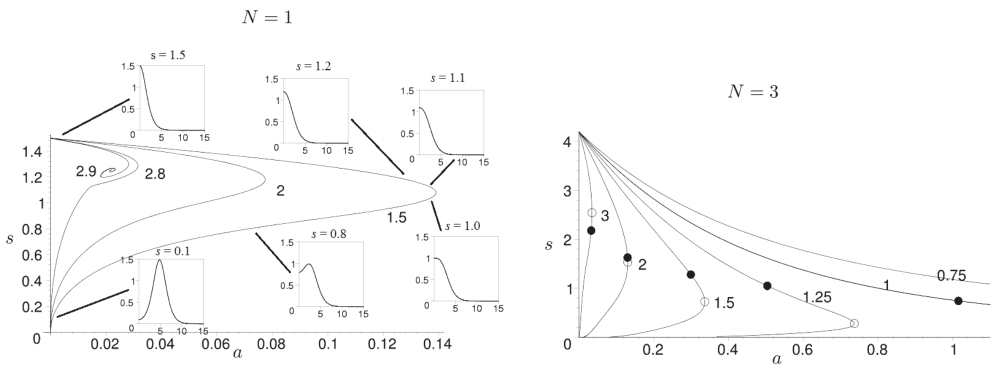
We remark that a global bifurcation theory is still an open question. However, the fact that $u > 0$ along the entire bifurcation curve is shown in Lemma 4 below.

The solution to (4) is not necessarily unique when $a \neq 0$: depending on parameter values, there can be other possible solutions that are nonmonotone and whose peak can be located far from the origin with $s$ near zero. For example, consider (4) with $N = 3, p = 2$. The bifurcation diagram $s = u(0)$ versus $a$ is computed numerically in Figure 2(b) for several different values of $q$. When $q > 1$, the bifurcation curve is bounded and there is a fold point at some $a = a_c$ beyond which there are no solutions. This fold point is precisely condition (S1). On the other hand, if $q \leq 1$, then a solution exists for all $a > 0$ with $s \to 0$ as $a \to \infty$. A typical steady state profile evolution along the bifurcation curve in one dimension is shown in Figure 2(a); note the "multibump" solutions along the lower part of the bifurcation branch. These are studied in detail using asymptotic methods in Appendix A.

The main goal of this section is to classify under which conditions on $p, q, N$ the bifurcation graph is bounded in the $(a, s)$ plane. The following theorem provides these conditions.

THEOREM 2. *Given $a \geq 0$, let $u(r)$ be a positive solution to* (4) *and let*

$$s := u(0). \tag{6}$$




(a) (b)

FIG. 2. *Bifurcation diagram for* (4) *of $a$ versus $s = u(0)$ with $p = 2$ and for several different values of $q$ as indicated.* (a) *$N = 1$. There is a fold point for all values of $q$. The bifurcation graph changes its topology at around $q = 2.8$ but is bounded for all $q$. The inserts show the profile of the steady state $u(r)$ for $q = 1.5, p = 2$ and for $s$ as indicated.* (b) *$N = 3$. Fold point is indicated by an empty circle. Nonradial instability threshold is indicated with a filled circle. If $q > 2.1$, then fold point instability dominates. If $q < 2.1$, then nonradial instability dominates. The fold point exists if $q > 1$; the bifurcation graph is unbounded if $q < 1$.*

*Define*

(7a) $$q_\star := \frac{N(p-1) - 2(p+1)}{2}; \qquad q^\star := \frac{(p-1)N}{2};$$

(7b) $$q_c := \frac{(p-1)(N-1)}{2}.$$

*The following holds:*

(i) *Suppose that $p \in (1, p^\star)$, where $p^\star$ is the critical exponent given by* (5) *and $q \geq 0$. Given any constant $a_0 > 0$, there exists a constant $s_0 = s_0(a_0, p, q)$ such that if $0 \leq a < a_0$, then the solution to* (4) *does not exist if $s > s_0$.*

(ii) *Suppose that either $N \geq 3$ and $q > q_c$ or else $N \leq 2$ and $q > q^\star$. There exists a constant $a_0$ such the positive solution to* (4) *does not exist if $a > a_0$.*

(iii) *If $N \geq 3$, $q_\star < q < q_c$, and $q \geq 0$, then the positive solution to* (4) *exists for all $a \geq 0$, provided that $1 < p < p^\star$.*

When (i) and (ii) simultaneously hold, the bifurcation graph in the positive $(a, s)$ plane is bounded. Note that $q_\star < 0$ iff $p < p_\star$ and moreover $q_\star < q_c < q^\star$. In particular, statements (i) and (ii) hold simultaneously in dimension $N \geq 3$ provided that $q > q_c$ and $p \in (1, p^\star)$; they hold in dimension $N = 1$ or 2 provided that $q > q^\star$ and $p > 1$. In conclusion, *the bifurcation curve is bounded whenever* (3) *is satisfied.* This proves the weaker version (S1*) of the key condition (S1).

*Remark* 1. We conjecture that the bifurcation curve exhibits a fold point whenever it is bounded, i.e., condition (S1) holds under conditions (3). As an example, consider Figure 2(b), where $N = 3, p = 2 < p^\star = 5$: according to Theorem 2, the bifurcation curve is bounded. Numerically, the fold point is observed whenever $q > 1 = q_c$. On the other hand the bifurcation curve is unbounded when $q \leq 1$; this is in agreement with statement (iii) of Theorem 2. In this case, numerics indicate that no fold point exists. (For the special case $N = 3, p = 2, q = 1$, the nonexistence of the fold point is rigorously proved in section 3.2.)

*Remark* 2. We think that $q^\star$ in (ii) can be replaced by $q_c$ and the condition $N \geq 3$ can be eliminated in (ii). However, we were unable to prove that.

*Remark* 3. We also conjecture that the condition $p < p^\star$ is not necessary in (iii); it is sufficient that $q_\star < q < q_c$ for (iii) to hold.

The proof of (ii) and (iii) of Theorem 2 is an immediate consequence of the following lemma.

LEMMA 3. *Consider the problem*

$$u'' + \frac{N-1}{r}u' - u + (\varepsilon + r^q)u^p = 0; \quad u'(0) = 0, \quad u > 0; \; u \to 0 \quad \textit{as} \quad r \to \infty. \tag{8}$$

*Suppose that* $1 < p < p^\star$, *and let* $q_\star, q^\star, q_c$ *be as given by* (7). *We have the following results:*

(i) *Suppose that* $q$ *satisfies*

$$q > q_c \quad \textit{if } N \geq 3 \textit{ or } q > q^\star \textit{ if } N \leq 2. \tag{9}$$

*Then there exists* $\varepsilon_0 = \varepsilon_0(p, q, N)$ *such that* (8) *has no solution for all* $0 \leq \varepsilon < \varepsilon_0$.

(ii) *Suppose that* $N \geq 3$ *and* $q = q_c$ *and* $\varepsilon = 0$. *Then* (8) *has no solution.*

(iii) *Suppose that* $N \geq 3$ *and* $q_\star < q < q_c$. *Then the solution to* (8) *exists for all* $\varepsilon > 0$. *Such solution is unique if* $\varepsilon = 0$.

We now give proofs of Theorem 2 and Lemma 3.

*Proof of Theorem* 2. We first show (i). First, suppose that $q \neq 0$. Consider the initial value problem

$$v_{rr} + \frac{N-1}{r}v_r - v + (1 + ar^q)v^p = 0, \qquad v'(0) = 0, \quad v(0) = s. \tag{10}$$

Rescale

$$v = sV; \quad r = \tau y,$$

where $\tau$ is to be specified. Then the equation for $V$ is

$$V_{yy} + \frac{N-1}{y}V_y - \tau^2 V + (\tau^2 s^{p-1} + a\tau^{q+2}s^{p-1}y^q)V^p = 0; \;\; V'(0) = 0, \;\; V(0) = 1. \tag{11}$$

Choosing $\tau = s^{-(p-1)/2}$, we then obtain

$$V_{yy} + \frac{N-1}{y}V_y + V^p - \varepsilon_1 V + \varepsilon_2 y^q V^p = 0; \;\; V'(0) = 0, \;\; V(0) = 1, \tag{12}$$

where

$$\varepsilon_1 = s^{-(p-1)}; \quad \varepsilon_2 = as^{-q(p-1)/2}. \tag{13}$$

Now consider the limiting problem

$$V_{0yy} + \frac{N-1}{y}V_{0y} + V_0^p = 0; \quad V_0(0) = 1, \; V_0'(0) = 0. \tag{14}$$

In Lemma 13 (see Appendix B) we show that for $p \in (1, p^\star)$, $V_0$ becomes negative at some $y = y_0$. In particular, there exists $y_1 > y_0$ and $C_1 > 0$ such that $v_0(y_1) < -C_1 < 0$. By continuity of solutions to the initial value problem with respect to parameters, $V$ can be made arbitrarily close to $V_0$ by choosing any sufficiently small $\varepsilon_1$,$\varepsilon_2$. In particular, there exists a $\varepsilon = \varepsilon(p, q) > 0$ such that for all $\varepsilon_1, \varepsilon_2 < \varepsilon$, we have $|V(y_1) - V_0(y_1)| < C_1/2 \implies V(y_1) < 0$ . Now given $a_0 > 0$ and for any $0 < a \leq a_0$, note that $\varepsilon_1, \varepsilon_2 < \varepsilon$ whenever $s > s_0$, where $s_0 := \max(\varepsilon^{-1/(p-1)}, (\varepsilon/a_0)^{-2/[q(p-1)]})$.

(The quantity $(\varepsilon/a_0)^{-2/[q(p-1)]}$ is interpreted to be zero if $q = 0$.) In this case, $v$ has a root and hence no solution to (4) exists when $a < a_0$ and $s > s_0$. This proves (i).

To prove (ii) we apply Lemma 3 after a change of variables $u \to a^{1/(1-p)}u$. Then (4) becomes (8) with $\varepsilon = 1/a$. Statement (i) of Lemma 3 immediately yields the desired result. The proof of (iii) follows from statement (iii) of Lemma 3. $\square$

*Proof of Lemma* 3. We start with the nonexistence results (i) and (ii) which are proved in Steps 1 to 4. Result (iii) is proved in Step 5.

*Step* 1. We first derive the following key identity:

$$\int_0^\infty r^{N-m} u^{p+1} \left[\varepsilon - c_1 r^q\right] dr > 0, \tag{15}$$

where

$$m = \begin{cases} 2, & N \geq 3, \\ 1, & N \leq 2, \end{cases} \qquad c_1 = \begin{cases} \frac{2}{(p-1)(N-1)}(q - q_c), & N \geq 3, \\ \frac{2}{(p-1)N}(q - q^\star), & N \leq 2. \end{cases}$$

In one and two dimensions, this is a consequence of Pohozhaev-type inequalities as we now show. First, multiply (8) by $r^{N-1}u$ and integrate by parts to obtain

$$-\int_0^\infty r^{N-1}u'^2 dr - \int_0^\infty r^{N-1}u^2 dr + \int_0^\infty r^{N-1}\left(\varepsilon + r^q\right)u^{p+1} dr = 0. \tag{16}$$

Next, multiply (8) by $r^N u'$ and integrate by parts to get

$$\begin{aligned} \left(-1 + \frac{N}{2}\right)\int_0^\infty r^{N-1}u'^2 dr + \frac{N}{2}\int_0^\infty r^{N-1}u^2 dr \\ - \frac{N+q}{p+1}\int_0^\infty r^{N-1+q}u^{p+1} dr - \varepsilon\frac{N}{p+1}\int_0^\infty r^{N-1}u^{p+1} dr = 0, \end{aligned} \tag{17}$$

where the boundary terms vanish by Lemma 11 of Appendix B. Combining (16) and (17) we obtain

$$\int_0^\infty r^{N-1}u^{p+1}\left[\varepsilon - \frac{2q - (p-1)N}{N(p-1)}r^q\right] dr = \frac{2(p+1)}{N(p-1)}\int_0^\infty r^{N-1}u'^2 dr.$$

This proves (15) in the case $N = 1, 2$. To obtain a sharper inequality for dimensions $N \geq 3$, we derive another identity as follows. Differentiating (8) with respect to $r$ we obtain

$$\frac{1}{r^{N-1}}\left(r^{N-1}u''\right)' - \frac{N-1}{r^2}u' - u' + \left(\varepsilon + r^q\right)pu^{p-1}u' + qr^{q-1}u^p = 0. \tag{18}$$

Multiplying (18) by $r^{N-1}u$, integrating on $[0, \infty]$, and using integration by parts we get

$$\begin{aligned} \int_0^\infty \Big\{ \left(u'r^{N-1}\right)' u' + (N-1)\, r^{N-3}uu' - r^{N-1}uu' \\ + \left(r^q + \varepsilon\right) r^{N-1}pu^p u' + qr^{q-1}u^{p+1}r^{N-1} \Big\} dr = 0. \end{aligned}$$

Using (8) and rearranging we obtain

$$\int_0^\infty r^{N-1}(p-1)\left(r^q + \varepsilon\right)u^p u' dr + q\int_0^\infty r^{N-2+q}u^{p+1} dr = \frac{(N-1)}{2}\int_0^\infty r^{N-3}\left(u^2\right)' dr. \tag{19}$$

Note that

$$\int_0^\infty r^{N-1+q} u^p u' dr = -\frac{N-1+q}{p+1} \int_0^\infty r^{N-2+q} u^{p+1} dr.$$

Thus we obtain
(20)

$$\int_0^\infty r^{N-2} u^{p+1} \left[ \varepsilon - \left( \frac{2q-(p-1)(N-1)}{(p-1)(N-1)} \right) r^q \right] dr = -\frac{1}{2} \frac{p+1}{p-1} \int_0^\infty r^{N-3} \left(u^2\right)' dr,$$

and moreover,

$$(21) \qquad -\int_0^\infty r^{N-3} \left(u^2\right)' dr = \left\{ \begin{array}{cc} (N-3) \int_0^\infty r^{N-4} u^2 dr, & N > 4 \\ u(0)^2, & N = 3 \end{array} \right. > 0.$$

This proves (15) for dimension $N \geq 3$.

*Step* 2. Given $q$ that satisfies (9), note that (15) holds with $c_1 > 0$. We now show that there exists a constant $C$ such that $u(0) > C\varepsilon^{-1/(p-1)}$ for all sufficiently small $\varepsilon$. Let $r_0 = (1/c_1)^{1/q} \varepsilon^{1/q}$ be the root of $\varepsilon - c_1 r^q = 0$. Then

$$\begin{aligned} \int_0^\infty r^{N-m} u^{p+1} \left[\varepsilon - c_1 r^q\right] dr &= \int_0^{r_0} r^{N-m} u^{p+1} \left[\varepsilon - c_1 r^q\right] dr \\ &\quad - \int_{r_0}^\infty r^{N-m} u^{p+1} \left[c_1 r^q - \varepsilon\right] dr > 0 \end{aligned}$$

so that

$$\begin{aligned} \int_0^{r_0} r^{N-m} u^{p+1} \left[\varepsilon - c_1 r^q\right] dr &> \int_{r_0}^\infty r^{N-m} u^{p+1} \left[c_1 r^q - \varepsilon\right] dr \\ &> \int_{r_1}^{r_1+r_0} r^{N-m} u^{p+1} \left[c_1 r^q - \varepsilon\right] dr \end{aligned}$$

for any $r_1 \geq r_0$. In particular, choose $r_1$ to satisfy $\varepsilon - c_1 r^q = -\varepsilon$, i.e., $r_1 = (2/c_1)^{1/q} \varepsilon^{1/q}$. Then $\varepsilon \geq \varepsilon - c_1 r^q$ on $[0, r_0]$ and $c_1 r^q - \varepsilon \geq \varepsilon$ on $[r_1, r_1 + r_0]$ so that

$$\int_0^{r_0} r^{N-m} u^{p+1} dr > \int_{r_1}^{r_1+r_0} r^{N-m} u^{p+1} dr.$$

It follows that $r^{N-m} u^{p+1}$ cannot be increasing on $[0, r_0 + r_1]$. In particular, $u$ cannot be increasing on $[0, C_1 \varepsilon^{1/q}]$, where $C_1 = (2/c_1)^{1/q} + (1/c_1)^{1/q}$. Now consider the initial value problem

$$(22) \qquad 0 = \hat{u}_{rr} + \frac{N-1}{r} \hat{u}_r - \hat{u} + \hat{u}^p (\varepsilon + r^q); \quad \hat{u}(0) = \xi, \quad \hat{u}'(0) = 0.$$

We claim that there exists a constant $C_2$ such that $\hat{u}$ is nondecreasing on the interval $[0, C_1 \varepsilon^{1/q}]$ whenever $\xi < C_2 \varepsilon^{-1/(p-1)}$. In fact, note that by the comparison principle, $\hat{u} < \xi v$, where $v$ solves $v_{rr} + \frac{N-1}{r} v_r - v = 0$, $v'(0) = 0, v(0) = 1$. It follows that $\hat{u} < \xi C_0$ on $[0, 1]$, where $C_0 \equiv v(1) > 1$ is some constant independent of $\varepsilon, p, q, \xi$. Now suppose that $u$ is increasing on $[0, r_m]$ and has a maximum at $r_m < C_1 \varepsilon^{1/q}$. At such a point,

$$\varepsilon + r_m^q = \frac{1}{\hat{u}^{p-1}} - \frac{u''(r_m)}{\hat{u}^p} \geq \frac{C_0^{1-p}}{\xi^{p-1}}.$$

It follows that

$$r_m \geq \left( \frac{C_0^{1-p}}{\xi^{p-1}} - \varepsilon \right)^{1/q} > C_1 \varepsilon^{1/q}$$

whenever

$$\xi < \frac{C_0^{-1}}{(C_1^q + 1)^{\frac{1}{p-1}} \, \varepsilon^{1/(p-1)}}.$$

Therefore $\hat{u}$ is increasing on $[0, C_1\varepsilon^{1/q}]$ whenever $\xi < C_2 \varepsilon^{-1/(p-1)}$, where $C_2 = \frac{C_0^{-1}}{(C_1^q+1)^{\frac{1}{p-1}}}$. It follows that $u(0) > C_2 \varepsilon^{-1/(p-1)}$.

*Step* 3. We claim that there exists a number $\xi_0$ such that for all $\varepsilon < 1$ and all $\xi > \xi_0$, the solution $\hat{u}$ to (22) crosses the $x$-axis. To see this, let

$$\hat{u} = \xi v; \quad r = \xi^{\frac{1-p}{q+2}} s.$$

Then (22) becomes

$$v_{ss} + \frac{(N-1)}{s} v_s + s^q v^p = \xi^{-\frac{2(p-1)}{q+2}} v - \varepsilon \xi^{\frac{q(p-1)}{q+2}} v^p; \quad v(0) = 1, \ v'(0) = 0. \tag{23}$$

Assume there is no such $\xi_0$ as required. Then there are $\xi_k \to \infty$ and $0 \leq \varepsilon_k \leq 1$ such that the solution of (23) with $\xi = \xi_k$ and $\varepsilon = \varepsilon_k$ is positive for $s > 0$. Define $\beta_k := \varepsilon_k \xi_k^{\frac{q(p-1)}{q+2}}$. After passing to a subsequence, we may assume that $\beta_k \to \beta$. We separate the argument into two parts.

*Case* 1. $\beta = \infty$. Let $t = \beta_k^{1/2} s$. Then (23) becomes

$$v_{tt} + \frac{(N-1)}{t} v_t + \frac{1}{\beta_k} \left( \beta_k^{-q/2} t^q v^p - \xi_k^{-\frac{2(p-1)}{q+2}} v \right) + v^p = 0 \tag{24}$$

with $v(0) = 1$ and $v'(0) = 0$. In the limit $k \to \infty$, (24) becomes

$$v_{tt} + \frac{(N-1)}{t} v_t + v^p = 0; \quad v(0) = 1, \ v'(0) = 0. \tag{25}$$

Now by Lemma 13 in Appendix B, the solution to (25) crosses zero, provided that $p < p^\star$. By continuity, it follows that the solution $v$ to (24) also crosses zero when $k$ is sufficiently large, which is a contradiction.

*Case* 2. $\beta < \infty$. In this case, the solution to (23) converges to the solution to

$$v_s + \frac{(N-1)}{s} v_s + s^q v^p + \beta v^p = 0; \quad v(0) = 1, \ v'(0) = 0. \tag{26}$$

By Lemma 13 in Appendix B, the solution to (26) crosses zero, provided $p^\star > p > 1$ and $q > q_\star$. By continuity, it follows that the solution $v$ to (23) also crosses when $k$ is large, which is a contradiction again. This proves the claim.

*Step* 4. Let $\varepsilon_0 = \min\{1, (\frac{C_2}{\xi_0})^{p-1}\}$. Suppose that there exists a solution to (8) with $\varepsilon < \varepsilon_0$. Then from Step 2, we have that $u(0) > \xi_0$. But then by Step 3, $u(x)$ will cross the $x$-axis, a contradiction to the assumption that $u > 0$ for all $x$. This concludes the proof of statement (i). To prove (ii), note that in the case $\varepsilon = 0, q = q_c$, the identity (20) reduces to $0 = -\int_0^\infty r^{N-3}(u^2)'$, which contradicts (21).

*Step* 5. We now discuss the existence results with $\varepsilon = 0$ and $N \geq 3$. If $p \in (1, p^\star)$ where $p^\star = \frac{N+2}{N-2}$ is the critical exponent, then the existence is an immediate consequence of a more general result proved in [2], whose statement we reproduce here for the reader's convenience. Namely, consider the more general problem

$$0 = u_{rr} + \frac{N-1}{r}u_r - u + u^p h(r). \tag{27}$$

Then Corollary 4.8 of [2] implies that a solution to (27) exists provided that $p \in (1, p^{\star})$ and $|h(r)| < C + r^q$ for some constant $C > 0, 0 < q < q_c$, for all $r \geq 0$. We remark that the necessary condition $q < q_c$ follows immediately from (15) with $\varepsilon = 0$; the condition $q_\star < q$ is the result of combining Pohozhaev identities (16), (17) with $\varepsilon = 0$,

$$\int_0^\infty r^{N-1}u^2 dr + \left(-1 + \frac{N}{2} - \frac{N+q}{p+1}\right)\int_0^\infty r^{N-1+q}u^{p+1} = 0,$$

so that $-1 + \frac{N}{2} - \frac{N+q}{p+1} > 0 \iff q_\star < q$.

Next we show uniqueness when $q \in (q_\star, q_c)$ and $\varepsilon = 0$. We follow the method outlined in [17], which works for more general equations of the form (27). Make a change of variables

$$u(r) = v(s)g(r),$$

where $s = s(r)$ is to be specified shortly. We have

$$\begin{aligned} u_r &= v_s \frac{ds}{dr} g + vg', \\ u_{rr} &= v_{ss}\left(\frac{ds}{dr}\right)^2 g + 2v_s g' \frac{ds}{dr} + v_s \frac{d^2 s}{dr^2} g + vg'' \end{aligned}$$

so that (27) becomes

$$v_{ss}\left(\frac{ds}{dr}\right)^2 g + v_s\left(2g'\frac{ds}{dr} + \frac{d^2s}{dr^2}g + \frac{N-1}{r}\frac{ds}{dr}g\right) + v\left(g'' + \frac{N-1}{r}g' - g\right) + v^p g^p h = 0.$$

Next choose $s$ so that

$$\frac{d^2 s}{dr^2} = -\frac{ds}{dr}\left(2\frac{g'}{g} + \frac{N-1}{r}\right)$$

so that

$$\frac{ds}{dr} = g^{-2}r^{-(N-1)}.$$

Also choose $g$ so that

$$\begin{aligned} g^p h &= \left(\frac{ds}{dr}\right)^2 g = g^{-3}r^{-2(N-1)}, \\ g &= h^{\frac{1}{-3-p}} r^{\frac{2(N-1)}{-3-p}}. \end{aligned}$$

We then get

$$v_{ss} + F(r)v + v^p = 0, \tag{28}$$

where

$$F(r) = \left(g'' + \frac{N-1}{r}g' - g\right)g^3 r^{2(N-1)}; \qquad g = h^{\frac{1}{-3-p}} r^{\frac{2(N-1)}{-3-p}}. \tag{29}$$

For (28), Theorem 1 of [17] guarantees uniqueness, provided that $F(r)$ satisfies the so-called $\Lambda$-property on $(0,\infty)$; that is, $F(r)$ has at most one maximum and no interior minimum. It remains to verify this property.

Note that

$$q_c - q_\star = \frac{p+3}{2}.$$

This suggests a change of variables,

$$\delta := (q_c - q)\,\frac{2}{p+3}.$$

Then

$$q \in (q_\star, q_c) \iff \delta \in (0,1) \tag{30}$$

and using $h = r^q$, $F(r)$ becomes

$$F(r) = -c_1 r^{2(-1+\delta)} - r^{2\delta}, \text{ where } \ c_1 := (N-1-\delta)(N-3+\delta)/4 > 0.$$

Provided that (30) holds, note that $F'(r) = -[2(\delta-1)c_1 + 2\delta r^2]r^{-3+2\delta}$ has a unique positive root at $r = \sqrt{c_1(1-\delta)/\delta}$ and $F(r)$ is increasing for small positive $r$. This shows that $F(r)$ has the $\Lambda$-property. Therefore Theorem 1 of [17] proves the uniqueness of solution to (8) with $\varepsilon = 0$ provided $q \in (q_\star, q_c)$. $\square$

Finally, we show that the entire bifurcation branch is positive.

LEMMA 4. *Consider the bifurcation curve in $(a,s)$ for the solution $u(r)$ to (4) where $s = u(0;a)$. Then $u > 0$ for all $s > 0$ along the bifurcation curve.*

*Proof.* First, suppose that $u(r)$ solves

$$u_{rr} + \frac{N-1}{r}u_r - u + |u|^p\,(1+ar^q) = 0, \qquad u'(0) = 0, \quad u \to 0 \text{ as } r \to \infty. \tag{31}$$

Moreover, suppose that $u(0) > 0$ and $a > 0$. We claim that $u(r) > 0$ for all $r \geq 0$. We proceed by contradiction: suppose that $u(r) < 0$ for some $r$. Then $u$ has a global minimum at some point $r_0$ with $u(r_0) < 0$. But then $u_{rr}(r_0) \geq 0$, $u_r(r_0) = 0$ so that $0 = u_{rr}(r_0) + \frac{N-1}{r}u_r(r_0) - u(r_0) + |u(r_0)|^p\,(1+ar_0^q) \geq -u(r_0) + |u(r_0)|^p\,(1+ar_0^q) > 0$. This shows that $u(r) \geq 0$ for all $r > 0$. To show that $u(r) > 0$, suppose that $u(r_0) = 0$ for some $r_0$ with $u(r) \geq 0$ elsewhere. Then $u'(r_0) = 0$, so by uniqueness of solutions to ODEs, $u = 0$ for all $r \geq 0$, contradicting $u(0) > 0$. This proves the claim.

Now consider the bifurcation curve $(a,s)$ except that $u^p$ is replaced by $|u|^p$ in (4). Then $u > 0$ along the bifurcation curve. But then $u$ also solves the original problem (4) and $u > 0$. $\square$

We remark that a sign-change solution may exist if the condition $u \to 0$ as $r \to \infty$ is dropped in (31).

Theorem 2 provides conditions for when the bifurcation curve is bounded. To obtain a more refined information, we examine what happens to the bifurcation curve when $u(0)$ is small. In this case, there may exist solutions to (4) which attain maximum far away from the origin. These are studied using formal asymptotics in Appendix A. In dimensions $N \geq 2$, this analysis also leads to the threshold $q = q_c$.

**3. Stability analysis.** We now study the stability of the time-dependent problem (1). It is convenient to consider a more general problem,

$$\begin{cases} u_t = \Delta u - u + u^p h(x;a)\dfrac{c_0}{\int_{\mathbb{R}^N} u^{p+1} h(x;a)dx}, & x \in \mathbb{R}^N, \\ \nabla u(0,t) = 0, \quad u \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{32}$$

where $h(x) = h(r;a)$ is a radially symmetric function depending on the parameter $a$; the model (1) corresponds to $h = 1 + ar^q$. The constant $c_0$ is chosen so that the time-independent solution is the ground state satisfying

$$(33)\quad u_{0rr} + \frac{N-1}{r}u_{0r} - u_0 + u_0^p h(r;a) = 0, \quad u_0'(0) = 0, \ \ u_0 \to 0 \text{ as } r \to \infty, \ \ u_0 > 0;$$

that is,

$$c_0 = \int_{\mathbb{R}^N} u_0^{p+1} h dx.$$

Since the constant $c_0$ can be scaled out by scaling $u$, its inclusion does not change the stability properties.

While some of the results (and derivations) below are valid for a more general function $h$, we do not attempt to state the most general version of our results and will simply use $h = 1 + aq^q$ whenever required. In particular the proof of Lemma 5 and therefore Theorem 6 which relies on it, makes explicit use of $h = 1 + ar^q$.

The condition $\nabla u(0,t) = 0$ will be necessary to avoid translational instabilities. Equivalently, we may simply restrict (32) to the positive quadrant $\Omega = \{(x_1, x_2, \ldots, x_N) : x_i > 0, i = 1 \ldots N\}$ and impose Neumann boundary conditions on $\partial\Omega$. In this setting, the spike solution at the center becomes a boundary spike at the corner of $\Omega$.

When $h = 1$, the problem (32) and its generalizations are sometimes referred to as the *shadow system* [28]. It naturally occurs in the high diffusivity ratio limit of some reaction-diffusion systems, for example, the Gierer–Meinhardt model [27] and the Gray–Scott model [20], [4]. The main feature of (32) with $h = 1$ is that the integral term in the denominator stabilizes the large eigenvalues [28].

We begin our investigation by linearizing around the steady state. Set

$$u(x,t) = u(r) + e^{\lambda t} Z(x),$$

where $u(r)$ satisfies (33) (here and below we drop the subscript $_0$ for convenience) and $Z \ll 1$. Define

$$LZ := \Delta Z - Z + u^{p-1} hpZ. \tag{34}$$

Then we have

$$\begin{cases} \lambda Z = LZ - u^p h \frac{(p+1)}{c_0} \int_{\mathbb{R}^N} Z u^p h dx. \\ \nabla Z(0) = 0; \quad Z \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{35}$$

In one dimension the condition $Z'(0) = 0$ ensures that $Z$ is even (i.e., radially symmetric) eigenfunction. In dimensions $N \geq 2$, the problem (35) has a radially symmetric eigenfunction but may also have nonradially symmetric modes. We start by studying radially symmetric perturbations.

**3.1. Radially symmetric perturbations.** In this section we examine the radial stability of (32). That is, we consider solutions $(Z, \lambda)$ to (35), where $Z$ is restricted to the space of radially symmetric functions. As before, let

$$s = u(0;a), \tag{36}$$

where $u(x;a)$ is the ground state solution to (33). We will also assume that

$$h(x) = 1 + a|x|^q; \qquad p \in (1, p^\star) \text{ if } N \geq 3 \text{ or } p > 1 \text{ if } N = 1 \text{ or } 2. \tag{37}$$

Then there is a unique value $s_0$ with $a = 0$ which corresponds to the unique ground state solution to (33) with $a = 0$ [17]. Now consider the bifurcation curve $(s, a(s))$

going through $s = s_0, \ a = 0$. Suppose that such curve has a fold point. Our main result here is to show condition (S3) in one dimension. In addition, we will show that the even eigenfunction at the fold point of (35) corresponding to a zero eigenvalue has a root; this will prove condition (S2). We start with the following lemma, which explicitly uses the form (37).

LEMMA 5. *Let $h = 1 + ar^q$ and $s = u(0; a)$, where $u(x; a)$ is the ground state solution to* (33). *Suppose the bifurcation curve $a = a(s)$ has the following properties:* (1) *$a(s_0) = 0$ for some $s_0$*; (2) *$a'(s_c) = 0$ for some $s_c$ and $a'(s) \neq 0$ for all $s \in (s_c, s_0]$. Then the following conditions are equivalent for $s \in [s_c, s_0]$:*

(i) *The local eigenvalue problem*

$$LZ = \lambda Z, \quad Z'(r) = 0, \quad Z(r) \to 0 \text{ as } r \to \infty, \tag{38}$$

*admits a zero eigenvalue $\lambda = 0$ corresponding to a radially symmetric eigenfunction $Z$.*

(ii) $\frac{\partial a}{\partial s} = 0$, *where $u_s = \partial u / \partial s$.*

(iii) $Lu_s = 0$.

*Proof.* Note that $u_s$ satisfies

$$Lu_s = -u^p h_a \frac{\partial a}{\partial s}. \tag{39}$$

It immediately follows that (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). The main difficulty is showing that (i) $\Longrightarrow$ (ii). For this, we will make use of the following identity:

$$Lu = u^p h \left(p - 1\right). \tag{40}$$

We proceed in three steps.

*Step* 1. Suppose that (38) admits a zero eigenvalue with a radial $Z(r)$ and with $\frac{\partial a}{\partial s} \neq 0$. We claim that $Z(r)$ has at least two positive roots. Multiplying (40) by $Z$ we obtain $\int_0^\infty Zu^p(1 + ar^q)r^{N-1}dr = 0$. It follows that $Z$ has at least one positive root. Let $r_1 > 0$ be the first root of $Z$. Multiplying (39) by $Z$, integrating by parts, we obtain that $\int_0^\infty Zu^p r^q r^{N-1}dr = 0$, where we used $h_a = r^q$ and $\frac{\partial a}{\partial s} \neq 0$. Taking a linear combination, we then obtain $\int_0^\infty (r_1^q - r^q)Zu^p r^{N-1}dr = 0$. Thus $Z$ must have a root other than $r_1$.

*Step* 2. Consider the problem

$$LY = 0; \quad Y(0) = 1, \quad Y'(0) = 0. \tag{41}$$

We claim that for $s \in (s_c, s_0]$, $Y$ has at most one positive root. First, note that the eigenvalue problem (38) has exactly one positive eigenvalue when $s = s_0, \quad a = 0$; the corresponding eigenfunction $Z(r)$ is radial and does not change signs. By the oscillation theorem, it follows that the solution $Y$ of (41) has precisely one zero when $s = s_0$. Next, suppose there exists $s \in (s_c, s_0]$ such that (41) has two roots. Since it is known that $Y$ has only one root when $s = s_0$, by continuity, there must exist $\bar{s}$ such that $Y$ has one positive root when $s \in [\bar{s}, s_0]$ but two roots for $s < \bar{s}$. Now consider a sequence $s_k \to \bar{s}$ with $s_k < \bar{s}$ and let $r_{1,k}$ and $r_{2,k}$ denote the two roots of $Y(r; s_k)$ with $r_{1,k} < r_{2,k}$. Let $r_1 = \limsup_{k\to\infty} r_{1,k}$ and let $r_2 = \limsup_{k\to\infty} r_{2,k}$. Then either $r_1 = r_2$ or $r_2 = \infty$. The former case implies that when $s = \bar{s}$, $Y(r_1) = Y'(r_1) = 0$; but then $Y(r) \equiv 0$, contradicting $Y(0) = 1$. Hence we have $r_2 = \infty$ so that $Y(r) \to 0$ as $r \to \infty$. But this implies that when $s = \bar{s}$, $Y$ is an eigenfunction satisfying (38)

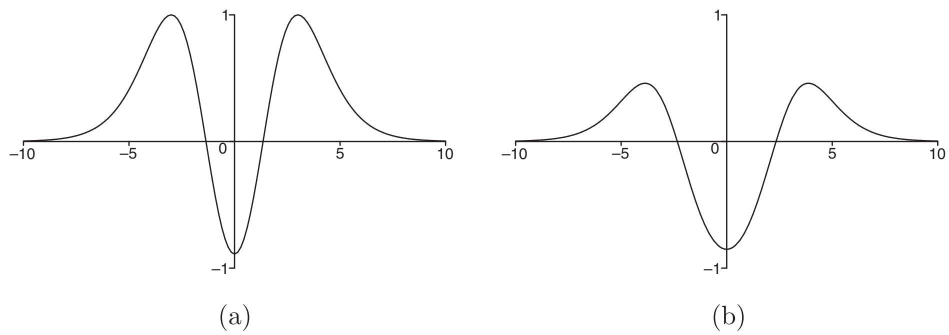



FIG. 3. (a) *The dimple eigenfunction at the fold point, corresponding to the zero eigenvalue of* (1) *with* $N = 1, p = 2, q = 2, a = 0.079$. *The shape of the eigenfunction is responsible for pulse replication.* (b) *The dimple eigenfunction for the reduced Gray–Scott model* (2), $N = 1$, *taken from* [14].

with $\lambda = 0$, having a unique positive root. By Step 1, this implies $\frac{\partial a}{\partial s} = 0$, which contradicts the assumption that $s \in (s_c, s_0]$.

*Step* 3. If $s \in (s_c, s_0]$ and $\lambda = 0$, Step 2 shows that $Z$ has at most a root. But this contradicts Step 1. ∎

We now state our main result for stability with respect to radially symmetric perturbations.

THEOREM 6. *Suppose that* $h$ *is as given in* (37) *and let* $s = u(0; a)$, *where* $u(x; a)$ *is the ground state solution to* (33). *Suppose the bifurcation curve* $a = a(s)$ *has the following properties:*

(i) $a(s_0) = 0$ *for some* $s_0$;

(ii) $a'(s_c) = 0$ *for some* $s_c$ *and* $a'(s) \neq 0$ *for all* $s \in (s_c, s_0]$.

*If* $s = s_c$, *then* (35) *admits a zero eigenvalue whose eigenfunction is given by* $Z = \frac{\partial u}{\partial s}|_{s=s_c}$. *Moreover,* $Z(r)$ *has at least one root* $r > 0$. *Thus condition* (S2) *is proven.*

*Let* $s \in (s_c, s_0]$. *By Lemma* 5, *the corresponding nonlocal eigenvalue problem* (35) *is stable with respect to radially symmetric perturbations.*

An example of the eigenfunction $\frac{\partial u}{\partial s}|_{s=s_c}$ with $N = 1$, $p = 2, q = 2$ is shown in Figure 3. The pulse splitting as observed in Figure 1(a) is due to its "upside-down Mexican hat" shape.

Note that Theorem 6 provides a partial generalization of [28], where the case $h = 1$ was proved.[1] Theorem 6 relies on the following lemma.

LEMMA 7. *Consider the local radially symmetric eigenvalue problem*

$$L\Phi = \lambda\Phi; \quad \Phi \text{ is radially symmetric} \tag{42}$$

*and the corresponding nonlocal problem,*

$$\lambda Z = LZ - u^p h \frac{(p+1)}{c_0} \int_{\mathbb{R}^N} Z u^p h dx; \quad Z \text{ is radially symmetric.} \tag{43}$$

[1] In [28], the stability of the problem

$$u_t = \Delta u - u + u^p \frac{1}{\int_{\mathbb{R}^N} u^m dx}$$

was considered; the case $h = 1$ in (32) corresponds to $m = p + 1$.

*Suppose* (42) *admits a unique positive eigenvalue. Then the nonlocal problem* (43) *is stable, i.e., it has no positive eigenvalues. Suppose* (42) *admits at least two positive eigenvalues. Then the nonlocal eigenvalue problem* (35) *is unstable, i.e., it admits at least one positive eigenvalue.*

*Proof.* Note that the eigenvalue problem (43) is self-adjoint so that the eigenvalues are all purely real. There are two cases to consider. First, suppose that $Z$ is an eigenfunction which satisfies (43) but does not satisfy (42); that is,

$$\int_{\mathbb{R}^N} Zu^p h dx \neq 0. \tag{44}$$

Then we may scale $Z$ so that (43) becomes

$$(L-\lambda)Z = u^p h; \qquad \int_{\mathbb{R}^N} Zu^p h dx = \frac{c_0}{p+1}. \tag{45}$$

Define

$$f(\lambda) := \int_{\mathbb{R}^N} (L-\lambda)^{-1} \left[u^p h\right] u^p h dx.$$

Then (45) becomes

$$f(\lambda) = \frac{c_0}{p+1}. \tag{46}$$

We compute

$$\begin{aligned} f'(\lambda) &= \int_{\mathbb{R}^N} (L-\lambda)^{-2} \left[u^p h\right] u^p h dx \\ &= \int_{\mathbb{R}^N} \left\{(L-\lambda)^{-1} \left[u^p h\right]\right\}^2 dx \end{aligned}$$

so that $f$ is always increasing. Also note that $f(\lambda)$ has a singularity at every positive eigenvalue of the local problem (42). Suppose that (42) admits $K$ positive eigenvalues, $K \geq 1$. Then $f(\lambda)$ has $K$ vertical asymptotes for positive $\lambda$. Now from (40) we note that

$$f(0) = \int_{\mathbb{R}^N} \frac{u}{p-1} u^p h dx = \frac{c_0}{p-1}$$

so that $f(0) > \frac{c_0}{p+1}$. Moreover, $f(\lambda) \to 0$ as $\lambda \to \infty$. Thus there are precisely $K-1$ positive solutions to (46).

We have shown that if $K \geq 2$, then (43) is unstable. It remains to show that (43) is stable when $K = 1$. Then there are precisely $K - 1 = 0$ positive solutions (46); hence there are no positive eigenvalues of (43) whose eigenfunction satisfies (44). It remains to consider the case $\int Zu^p h = 0$; $\quad K = 1$. But then $Z$ satisfies $LZ = \lambda Z$. Thus $\lambda = \lambda_1$, where $\lambda_1$ is the unique positive eigenvalue of (42). Now multiplying (40) by $Z$ and integrating, we then obtain $\lambda_1 \int uZ = 0$. Since we assumed $\lambda_1 \neq 0$, and $u > 0$, this means that $Z$ must change sign. But this contradicts the fact that $Z$ is the eigenfunction of the principal eigenvalue of the local problem (42). ∎

*Proof of Theorem* 6. First, note that when $a = 0$, $s = s_0$, we have $h(x) = 1$. In this case, the problem $LZ = 0$ admits $N$ independent solutions given by $Z_k = \hat{e}_k u'(r), k = 1 \ldots N$, where $\hat{e}_k$ is the $k$th unit vector and $u(r)$ is the radially symmetric ground state solution to (33) with $h = 1$. Thus the local eigenvalue problem $LZ = \lambda Z$ admits $N$ eigenfunctions corresponding to a zero eigenvalue. Moreover, it is well known that $u(r)$ is unique and is a decreasing function [17]. It follows that the nodal set $\{x : Z_k = 0\}$

$\varepsilon\,\{x : x_k = 0\}$, which divides $\mathbb{R}^N$ into exactly two connected sets. By the oscillation theorem there must be a positive eigenvalue whose eigenfunction has no root. Such an eigenvalue is unique and the corresponding eigenfunction is radially symmetric; all other radially symmetric eigenfunctions correspond to strictly negative eigenvalues. This proves that (42) admits precisely one positive eigenvalue when $s = s_0$. Next, note that the eigenvalues are all real since (43) is self-adjoint. By Lemma 5, the eigenvalues cannot be zero for $s \in (s_c, s_0)$. By continuity it follows that (42) admits exactly one positive eigenvalue for all $s \in (s_c, s_0]$. By Lemma 7, it then follows that (43) is stable.

We now prove that $u_s = \partial u/\partial s$ is an eigenfunction of (43) corresponding to $\lambda = 0$ whenever $s = s_c$. Certainly $Lu_s = 0$ (see Lemma 5). We now show that

$$\int_{\mathbb{R}^N} u_s u^p h dx = 0 \tag{47}$$

so that $u_s$ is indeed an eigenfunction of (43) corresponding to $\lambda = 0$. This follows by multiplying the identity (40) by $u_s$ and then integrating by parts and using $Lu_s = 0$. Equation (47) also shows that $u_s$ has a strictly positive root since $h, u > 0$. □

**3.2. Nonradial perturbations in three dimensions.** Theorem 6 shows that the top branch of the bifurcation curve is stable with respect to radially symmetric perturbations. This implies full stability in one dimension. However, in higher dimensions, nonradial instabilities can and do occur. This study considers such instabilities in three dimensions. As before, the starting point is the eigenvalue problem (35). We then use spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$
$$\Delta Z = Z_{rr} + \frac{2}{r}Z_r + \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}Z_{\phi\phi} + \frac{1}{\sin\theta}\left(\sin\theta Z_\theta\right)_\theta\right).$$

We decompose the eigenfunction as

$$Z(x,y,z) = \Phi(r)Y_l^m(\theta,\phi);\ l = 0, 1, \dots; \quad m = 0, \pm 1 \dots \pm l,$$

where $Y_l^m$ are the spherical harmonics (see, for example, Chapter 10 of [26]). Now note that $Y_0^0 = 1$ so that by the orthogonality property of spherical harmonics, we have $\int Y_l^m = 0$, $l \geq 1$, and $\int hZu^{p-1} = 0$. In particular the nonlocal term in (35) disappears so that $\Phi$ satisfies

$$\lambda_l\Phi = \Phi_{rr} + \frac{2}{r}\Phi_r - \frac{\gamma}{r^2}\Phi + phu^{p-1}\Phi; \qquad \gamma = l(l+1),\ l \geq 1.$$

Note that the case $l = 0$ corresponds to the radially symmetric eigenfunctions whose stability was already characterized by Theorem 6. The case $l = 1$ corresponds to translational modes; in such a case $Y_1^m = x/r, y/r$ or $z/r$. In particular, if $l = 1$, $h = 1$, then the solution is $\lambda_1 = 0$, $\Phi = u_r$. In general, $\lambda_1$ is typically unstable. It is for this reason that we have imposed the condition $\nabla u(0,t) = 0$ in (32); the translational modes $l = 1$ are inadmissible (they do not satisfy $\nabla Z(0) = 0$). Thus we need to only consider the stability of nonradial nodes $l \geq 2$. To get some insight, let us consider the case $h = 1 + ar^q$ with $q \geq q_c$, where $q_c$ is given in (7b). In Appendix A we have constructed a ring-like solution with $s = u(0) \to 0$, either for $q = q_c$ or $q > q_c$. Such solutions have the form

$$u(r) \sim Cw(y) \text{ where } y = r - r_0, \quad r_0 \gg 1,$$

where $C = (ar_0^q)^{1/(1-p)}$ and $w(y)$ is the one-dimensional ground state that satisfies (50). Since $w$ decays exponentially away from $r_0$, to leading order we have $\frac{2}{r}\phi_r - \frac{\gamma}{r^2}\phi \sim O(\frac{1}{r_0})$ so that

$$\lambda_l \phi \sim \phi_{yy} - \phi + pw^{p-1}\phi. \tag{48}$$

It is well known that (48) admits a positive eigenvalue (in fact, it is a special case of (42) with $N = 1$ and $h = 1$). This proves that $\lambda_l > 0$ for $l \geq 2$ if $u(0)$ is sufficiently small. In particular, as the bifurcation curve is traversed in the direction of decreasing $s$, the mode $l = 2$ eventually becomes unstable. This is illustrated in Figure 2(b).

Due to ordering principle for the local eigenvalue problem $LZ = \lambda Z$, the eigenvalues are ordered $\lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \cdots$. However, no such ordering exists between the radial eigenvalue $\lambda_r$ and $\lambda_2$, since $\lambda_r$ satisfies the *nonlocal* problem (35). This leads to the following question: *As the bifurcation curve is traversed starting with* $a = 0, u(0) = O(1)$, *can the nonradial mode* $\lambda_2$ *become unstable before the radial mode* $\lambda_r$*?* Since $\lambda_r$ becomes unstable at the fold point, the answer is yes provided that the bifurcation curve has no fold point. In particular, if the solution to (4) is unique for all $a > 0$, then the fold point does not exist. We now show that this is the case when $p = 2$ and $q = q_c = 1$. Using Theorem 1 of [17], the solution is unique if the function $F(r)$ given by (29) with $h(r) = 1 + ar$ satisfies the $\Lambda$ property (as described below (29)). After some algebra we simplify to obtain

$$F(r) = -r^{-6/5}(1+ar)^{-14/5}\left(r^4a^2 + 2r^3a + r^2 + \frac{2}{5}ar + \frac{4}{25}\right),$$

$$F'(r) = \frac{-2}{125}r^{-6/5}(1+ar)^{-14/5}\left(25r^4a^2 + 50r^3a - \left(75a^2 - 50\right)r^2 - 45ar - 12\right).$$

Now clearly, $F \to -\infty$ as $r \to 0^+$. To show the $\Lambda$ property, it suffices to show that $F' = 0$ has a unique solution. But this follows from the Descartes rule of signs, since the coefficients in the polynomial inside $F'(r)$ change sign precisely once.

To summarize, in the case $p = 2, q = q_c = 1$, the radial mode $\lambda_r$ is stable for all $a > 0$; however, the nonradial mode $\lambda_2$ becomes unstable for sufficiently large $a$.

When $p = 2, q > 1$, the bifurcation curve has a fold point, where $\lambda_r = 0$. In general it is unknown whether $\lambda_2$ becomes unstable before $\lambda_r$ or vice versa, as $a$ is increased. However, if $p = 2$ and $q$ is close to 1, then because of continuous dependence on parameters, $\lambda_2$ is destabilized before $\lambda_r$ as $a$ is increased. Numerically, we observe that the opposite is true if $q$ is sufficiently large, as the following two tables illustrate:

| $p=2, q=1.3$ | | | |
|---|---|---|---|
| $a$ | $s$ | $\lambda_r$ | $\lambda_2$ |
| 0.0000 | 4.1895 | −0.79 | −1.03 |
| 0.1104 | 3.1895 | −0.62 | −1.02 |
| 0.2311 | 2.2895 | −0.44 | −0.67 |
| 0.4410 | 1.1395 | −0.18 | −0.02 |
| *0.4523* | *1.0895* | −0.17 | *0.00* |
| 0.6044 | 0.3895 | −0.005 | 0.59 |
| **0.6046** | **0.3395** | **0.005** | 0.65 |
| 0.5981 | 0.2895 | 0.014 | 0.71 |
| 0.4370 | 0.0895 | 0.026 | 0.98 |
| 0.1647 | 0.001 | 0.0067 | 1.19 |

| $p=2, q=3$ | | | |
|---|---|---|---|
| $a$ | $s$ | $\lambda_r$ | $\lambda_2$ |
| 0.0000 | 4.1895 | −0.79 | −1.037 |
| 0.0183 | 3.6395 | −0.54 | −0.99 |
| 0.0343 | 2.5895 | −0.024 | −0.3 |
| **0.0344** | **2.5395** | **0.0015** | −0.27 |
| 0.0343 | 2.4895 | 0.027 | −0.23 |
| *0.0326* | *2.1895* | 0.18 | *0.00* |
| 0.0314 | 2.0895 | 0.23 | 0.066 |
| 0.0229 | 1.6395 | 0.42 | 0.39 |
| 0.0128 | 1.1395 | 0.46 | 0.66 |
| 0.0003 | 0.001 | 0.033 | 1.19 |

For $p = 2$ and a given $q$, these two tables list the values of $\lambda_r$ and $\lambda_2$, as well as $a = a(s)$, computed numerically. Starting with $a = 0 \implies s = 4.1895$, we followed the bifurcation curve in the direction of decreasing $s$. When $q = 1.3$, the fold point occurs at $a \sim 0.6046$; numerics confirm that the radial node $\lambda_r$ crosses zero at that point (see also Theorem 6). However the *nonradial* mode $\lambda_2$ becomes unstable at around $a \sim 0.4523$ on the top branch of the bifurcation curve. Hence in this case, the mode $\lambda_2$ becomes unstable *before* $\lambda_r$ as $a$ is increased from $a = 0$. When $q = 3$, the opposite behavior is observed: the fold point occurs at $a \sim 0.0344$, whereas the nonradial mode $\lambda_2$ is destabilized only on the bottom branch of the bifurcation curve. In particular the top branch of the bifurcation curve is stable with respect to $\lambda_2$ (and hence, stable with respect to all nonradial perturbations due to the ordering property). This is also illustrated in Figure 2(b), where the bifurcation curve is plotted along the threshold values of $a$ when $\lambda_r = 0$ or when $\lambda_2 = 0$ for several different values of $q$ with $p = 2$.

**4. Discussion.** In this paper, we have shown that even a single PDE with heterogeneity has a self-replication structure similar to that of more complicated reaction-diffusion systems, such as Gray–Scott. For our simpler model, we are able to prove *analytically* Nishiura–Uyama conditions (S1$^*$) and—under an additional hypothesis that (S1) also holds—conditions (S2) and (S3). These conditions are believed to be responsible for the initiation of the fully nonlinear self-replication process. The process itself and the ensuing dynamics are still very poorly understood. Nishiura–Uyama conditions are based on the steady state and its linearization; as such, they provide little information about the fully nonlinear self-replication dynamics.

In the Gray–Scott model, peanut splitting is the dominant self-replication mechanism in two dimensions as observed by [23], [19], [20]. On the other hand, it was observed numerically in [15] that either the radial or the peanut-type instability can be dominant in the Gierer–Meinhardt model in two dimensions, depending on parameter values. Our simplified model has a similar structure: either instability is possible, depending on how the parameters $p, q$ are chosen.

We conclude with the following conjecture, which is a generalization of Corollary 4.8 in [2].

Conjecture 8. *Consider the system*

$$0 = \Delta u - u + u^p h(r); \quad u > 0,\ u \to 0 \text{ as } r \to \infty. \tag{49}$$

*Suppose $p > 1$ and $h(r)$ satisfiy*

$$|h(r)| \le C\,(1 + r^q)\,, \quad \text{where } q \ge 0 \text{ and } q \in (q_\star, q_c),$$

*where $C$ is some constant and $q_\star, q_c$ are given by* (7). *Then* (49) *has a radially symmetric solution.*

In [2, Corollary 4.8], this result was shown under a more restrictive assumption $p \in (1, p^\star)$, in which case $q_\star < 0$. Here, we do not assume that $p < p^\star$; this assumption is replaced with the more general assumption $q > q_\star$.

**Appendix A. Asymptotic analysis of (4) with small $u(0)$.** We now examine the behavior of the solution with small $u(0)$. The goal is to use asymptotic methods to construct radially symmetric solutions concentrating on a ring of a large radius. Below we will determine the asymptotic magnitude of such a radius. The analysis is different for $N = 1$ or $N \ge 2$.

**One dimension.** We consider (4) with $N = 1$, in the limit $a \ll 1$:

$$u_{xx} - u + u^p(1 + ax^q) = 0; \quad a \ll 1; \quad u'(0) = 0; \quad u > 0; \quad u \to 0 \text{ as } x \to \infty.$$

We seek solutions of the form

$$u(x) \sim w\,(y) + R(x); \quad y = x - x_0; \quad x_0 \gg 0, \quad R \ll 1,$$

where $w(y)$ is the (unique) one-dimensional ground state of the homogeneous problem,

$$w_{yy} - w + w^p = 0; \quad w'(0) = 0, \quad w > 0, \quad w \to 0 \text{ as } |y| \to \infty, \tag{50}$$

and $R$ is the small remainder term. Then $R$ satisfies

$$R_{yy} - R + pw^{p-1}R + ax^q w^p = 0. \tag{51}$$

Note also that

$$\left(w_y\right)_{yy} - w_y + pw^{p-1}w_y = 0. \tag{52}$$

Multiplying (51) by $w_y$, integrating from $-x_0$ to $\infty$, and using (52), we get

$$\left(R_y w_y - R w_{yy}\right)|_{-x_0}^{\infty} + a\int_{-x_0}^{\infty} (y + x_0)^q w_y w^p dy = 0.$$

Since $w$ decays exponentially as $|y| \to \infty$, we can replace $\int_{-x_0}^{\infty}$ by $\int_{-\infty}^{\infty}$. Using integration by parts we estimate

$$\begin{aligned}\int_{-x_0}^{\infty} (y + x_0)^q w_y w^p dy &\sim -\int_{-\infty}^{\infty} \frac{1}{p+1} w^{p+1} q\,(y + x_0)^{q-1}\, dy \\ &\sim -\frac{q}{p+1} x_0^{q-1} \int_{-\infty}^{\infty} w^{p+1}.\end{aligned}$$

Now for small $x$, we have that $R_{xx} - R \sim 0$ and $w \sim C_0 e^{-|x-x_0|}$. The constant $C_0$ is obtained by expanding $w$ in the far-field $|y| \to \infty$. Thus we have

$$w \sim C_0 e^{-x_0} e^x; \qquad R \sim C_1 e^x + C_2 e^{-x}; \qquad x \sim 0.$$

Since $R$ must remain small as $x$ is increased, it follows that $C_1 = 0$. Moreover, $\left(R_x + w_x\right)_{x=0} = 0$, which implies $C_2 = C_0 e^{-x_0}$. We therefore obtain

$$\left(R_y w_y - R w_{yy}\right)_{y=-x_0} = 2C_0^2 e^{-2x_0}.$$

This yields the following formula for $x_0$:

$$2C_0^2 e^{-2x_0} \sim a\frac{q}{p+1} x_0^{q-1} \int_{-\infty}^{\infty} w^{p+1} dy; \quad a \ll 1, x_0 \gg 1. \tag{53}$$

In case $p = 2$, we have $w(y) = \frac{3}{2}\operatorname{sech}^2(y/2)$ and $C_0 = 6$, $\int w^3 dy = 36/5$, so that

$$\frac{e^{-2x_0}}{x_0^{q-1}} \sim a\frac{q}{30}; \qquad p = 2. \tag{54}$$

In case $p = 3$, we have $w(y) = \sqrt{2}\operatorname{sech}(y)$ and $C_0 = 2\sqrt{2}$, $\int w^3 dy = \pi\sqrt{2}$, so that

$$\frac{e^{-2x_0}}{x_0^{q-1}} \sim aq\frac{\pi\sqrt{2}}{64}; \qquad p = 3. \tag{55}$$

**Ring solutions in a higher-dimension generic case.** We consider (4) with $N \geq 2$ in the limit $a \ll 1$. It is convenient to set

$$\varepsilon := a^{1/q}$$

so that (4) becomes

$$0 = u_{rr} + \frac{N-1}{r} u_r - u + u^p (1 + (\varepsilon r)^q). \tag{56}$$

The expansion we use is

$$r = \frac{1}{\varepsilon} r_0 + y; \quad u = U_0(y) + \varepsilon U_1(y) + \cdots .$$

Expanding to two orders we obtain

$$a r^q = (r_0 + \varepsilon y)^q = r_0^q + \varepsilon q r_0^{q-1} y + \cdots ,$$

$$0 = U_{0yy} - U_0 + (1 + r_0^q) U_0^p, \tag{57}$$

$$0 = U_{1yy} - U_1 + \frac{(N-1)}{r_0} U_{0y} + (r_0^q + 1)\, p U_0^{p-1} U_1 + U_0^p q r_0^{q-1} y. \tag{58}$$

Multiply (58) by $U_{0y}$ and integrate by parts; using (57) we obtain

$$\frac{q r_0^q}{(p+1)} \int_{-\infty}^{\infty} U_0^{p+1} dy = (N-1) \int_{-\infty}^{\infty} U_{0y}^2 dy. \tag{59}$$

The integrals can be further eliminated using Pohazhaev-type identities. Namely, multiply (57) by $U_0$ and integrate to get

$$-\int_{-\infty}^{\infty} U_{0y}^2 dy - \int_{-\infty}^{\infty} U_0^2 dy + (1 + r_0^q) \int_{-\infty}^{\infty} U_0^{p+1} dy. \tag{60}$$

Multiply (57) by $y U_{0y}$ and integrate to obtain

$$-\frac{1}{2} \int_{-\infty}^{\infty} U_{0y}^2 dy + \frac{1}{2} \int_{-\infty}^{\infty} U_0^2 dy - (1 + r_0^q) \int_{-\infty}^{\infty} \frac{U_0^{p+1}}{p+1} dy = 0. \tag{61}$$

Combining (60) and (61) we obtain

$$-2 \int_{-\infty}^{\infty} U_{0y}^2 dy + (1 + r_0^q) \int_{-\infty}^{\infty} U_0^{p+1} \left(1 - \frac{2}{p+1}\right) dy = 0. \tag{62}$$

Substituting (62) into (59) we finally obtain

$$r_0^q = \frac{(N-1)(p-1)}{2q - (N-1)(p-1)}. \tag{63}$$

The solution to (63) exists provided that

$$q > q_c = \frac{(N-1)(p-1)}{2}. \tag{64}$$

This is consistent with thresholds derived in Theorem 2 for the case $N \geq 3$. In particular, it is in agreement with the bifurcation diagram shown in Figure 2(b): for $q > q_c$, the curve approaches $a \to 0$ as $s \to 0$.

**Ring solutions in dimension $N = 3$, threshold case $p = q+1$.** The analysis is much more involved. For simplicity, we consider only the case $p = 2$. However, the result generalizes without difficulty for any $p > 1$. We summarize the result as follows.

PROPOSITION 9. *Suppose $N = 3$, $p = 2$, and $q = 1$. In the limit $a \gg 1$, let $r_0 \gg 1$ be the large solution to the equation*

$$a = \frac{1}{30} r_0^{-2} \exp(2r_0); \quad a, r_0 \gg 1.$$

*Then there exist solutions of* (4) *of the form*

$$u(r) \sim \frac{1}{r_0 a} w(r - r_0).$$

*Proof of Proposition* 9. We rescale

$$u(r) = \frac{1}{r_0 a} U(r)$$

and define

$$\varepsilon = \frac{1}{a r_0}$$

so that

$$0 = U_{rr} + \frac{2}{r} U_r - U + U^2 \left( \varepsilon + \frac{r}{r_0} \right). \tag{65}$$

The main idea is to separately solve the equation on $[0, r_0]$, then on $[r_0, \infty)$. Then $\varepsilon$ will be determined by requiring that $U(r_0^-) = U(r_0^+)$. So we treat (65) as two separate equations to solve: the first on $[0, r_0]$ with boundary conditions $U'(0) = 0 = U'(r_0)$ and the second on $[r_0, \infty)$ with boundary conditions $U'(r_0) = 0 = U'(\infty)$.

It will be shown below that $\varepsilon = O(r_0 e^{-2r_0})$. Therefore we will need to expand in both $\varepsilon$ and $\frac{1}{r_0}$. First, we treat $r_0$ as constant with respect to $\varepsilon$ and expand

$$U = U_0 + \varepsilon U_1 + \cdots .$$

We get

$$\begin{aligned} 0 &= U_{0rr} + \frac{2}{r} U_{0r} - U_0 + U_0^2 \frac{r}{r_0}, \\ 0 &= U_{1rr} + \frac{2}{r} U_{1r} - U_1 + 2U_0 U_1 \frac{r}{r_0} + U_0^2. \end{aligned}$$

Next we let

$$y = r - r_0$$

and expand

$$U_0(r) = U_{00}(y) + \frac{1}{r_0} U_{01}(y) + \frac{1}{r_0^2} U_{02}(y) + \cdots .$$

We have

$$(U_{00})_{yy} - U_{00} + U_{00}^2 = 0; \quad U_{00}'(0) = 0,$$

so that

$$U_{00}(y) = w(y).$$

At the next order we get

$$LU_{01} + 2w_y + yw^2 = 0, \tag{66}$$

where

$$L\phi := \phi_{yy} - \phi + 2w\phi.$$

Note that $L(yw) = yw^2 + 2w_y$ so that the solution to (66) is given by

$$U_{01} = -yw + Cw_y.$$

To determine the constant $C$ we impose the condition $U_{01}'(0) = 0$, which yields $C = -2$,

$$U_{01} = -yw - 2w_y.$$

Therefore $U_{01}$ is odd and at the next order we get

$$LU_{02} = f(y), \tag{67}$$

where $f(y)$ is a purely even function. Again, we treat this as two equations, one to the left and another to the right of $r_0$. To the left of $r_0$, multiply (67) by $w_y$ and integrate $y = -r_0 \dots 0$. We then get

$$(w_y U_{02y} - w_{yy} U_{02})|_{y=-r_0}^{y=0^-} \sim \int_{-\infty}^{0} f(y) w_y dy = -\int_0^\infty f(y) w_y dy. \tag{68}$$

To the right of $r_0$ we get

$$(w_y U_{02y} - w_{yy} U_{02})|_{y=0^+}^{y=\infty} \sim \int_0^\infty f(y) w_y dy. \tag{69}$$

Adding (68) and (69) together we get

$$w_{yy}(0)\left[U_{02}(0^+) - U_{02}(0^-)\right] = (w_y U_{02y} - w_{yy} U_{02})|_{y=-r_0}. \tag{70}$$

Therefore we need to determine the behavior near $r = 0$. Recalling that $y = r - r_0$ we write

$$w \sim C_0 e^r, \quad r \sim 0; \qquad C_0 = 6e^{-r_0}. \tag{71}$$

Since the solution decays near zero, we have $u^2 \ll u$ so that for small $r$

$$u_{rr} + \frac{2}{r} u_r - u \sim 0, \ u'(0) = 0.$$

Such a solution is given by

$$u = A \frac{e^r - e^{-r}}{r}, \tag{72}$$

where the constant $A$ is to be determined. To do so, we rewrite $U_{00} + \frac{1}{r_0} U_{01}$ as

$$\begin{aligned} U_{00} + \frac{1}{r_0} U_{01} &= w + \frac{1}{r_0}\left(-2w_y - yw\right) \\ &\sim \frac{C_0}{r_0} e^r \left(-2 - r + 2r_0\right). \end{aligned}$$

Evaluating at $r = r_0$, we obtain

$$\left(U_{00} + \frac{1}{r_0} U_{01}\right)\Big|_{r=r_0} \sim \frac{C_0}{r_0} e^{r_0} \left(-2 + r_0\right). \tag{73}$$

On the other hand, from (72) we estimate

$$u(r_0) \sim \frac{A}{r_0} e^{r_0}. \tag{74}$$

Matching (73) and (74) we obtain

$$A = C_0 \left(r_0 - 2\right). \tag{75}$$

Therefore the uniform expansion of $u$ is given by

$$u \sim w + \frac{1}{r}\left(-2w_y - yw\right) - C_0 \left(r_0 - 2\right) \frac{e^{-r}}{r}. \tag{76}$$

We now match decaying mode of (72) to the remainder of $U_0$ in the outer region:

$$-A \frac{e^{-r}}{r} \sim \frac{U_{02}}{r_0^2}.$$

This gives the following behavior of $U_{02}$ in the outer region:

$$U_{02} \sim C_0 \left(2 - r_0\right) \frac{r_0^2}{r} e^{-r} \tag{77}$$

$$\sim C_0 \left(2 - r_0\right) r_0 e^{-r_0} e^{-y}. \tag{78}$$

Using this we evaluate

$$\left(w_y U_{02y} - w_{yy} U_{02}\right)\big|_{y=-r_0} \sim -2C_0^2 \left(r_0 - 2\right) r_0. \tag{79}$$

Substituting (79), (71), and $w_{yy}(0) = -\frac{3}{4}$ into (70) we obtain

$$U_{02}(0^+) - U_{02}(0^-) \sim -96 e^{-2r_0} \left(r_0 - 2\right) r_0. \tag{80}$$

This yields

$$U_0(0^+) - U_0(0^-) \sim -96 e^{-2r_0} \left(1 - \frac{2}{r_0}\right). \tag{81}$$

Next we compute the jump in $U_1$. We expand

$$U_1 = U_{10}(y) + \frac{1}{r_0} U_{11}(y) + \cdots. \tag{82}$$

The leading order is

$$LU_{10} + w^2 = 0.$$

Imposing $U'_{10}(0) = 0$ and recalling that $Lw = w^2$, we get

$$U_{10}(y) = -w.$$

The next order then becomes

$$LU_{11} = 2w_y + 2yw^2.$$

Multiplying by $w_y$ and integrating to the left of $r_0$ we therefore get

$$(83) \qquad (w_y U_{11y} - w_{yy} U_{11})_{y=-r_0}^{0^-} \sim \int_{-\infty}^{0} \left(2w_y + 2yw^2\right) w_y dy = -\frac{6}{5},$$

and similarly to the right of $r_0$,

$$(84) \qquad (w_y U_{11y} - w_{yy} U_{11})_{0^+}^{\infty} = \int_{0}^{\infty} \left(2w_y + 2yw^2\right) w_y dy = -\frac{6}{5}.$$

Adding (83), (84) together and ignoring the exponentially small boundary terms we obtain

$$U_{11}(0^+) - U_{11}(0^-) \sim \frac{16}{5}$$

so that

$$(85) \qquad U_1(r_0^+) - U_1(r_0^-) \sim \frac{16}{5r_0}.$$

Putting together (81) and (85) we have

$$\begin{aligned} u(r_0^+) - u\left(r_0^-\right) &\sim \left(U_0(r_0^+) - U_0(r_0^-)\right) + \varepsilon\left(U_1(r_0^+) - U_1(r_0^-)\right) \\ &\sim -96e^{-2r_0}\left(1 - \frac{2}{r_0}\right) + \frac{\varepsilon}{r_0}\frac{16}{5}. \end{aligned}$$

The solvability condition is that this quantity is zero, that is,

$$\varepsilon \sim 30 r_0 e^{-2r_0}\left(1 - \frac{2}{r_0}\right).$$

This completes the proof. ◻

## Appendix B. Analysis of $u'' + \frac{N-1}{r}u' + \alpha r^q u^p + \beta u^p = 0$.

LEMMA 10. *Let $u$ satisfy*

$$u'' + \frac{N-1}{r}u' + g(r)u = 0, \ u > 0 \ \textit{on} \ [R, \infty)$$

*for some $R \geq 0$. Then for any $\delta > 0$, there exists $r_k \to \infty$ such that*

$$g(r_k) < \left(\frac{(N-2)^2}{4} + \delta\right) r_k^{-2}.$$

*Proof.* Assume there is $R_1 > R$ such that $g(r) \geq (\frac{(N-2)^2}{4} + \delta)r^{-2}$ for $r \geq R_1$. The equation

$$v'' + \frac{N-1}{r}v' + \left(\frac{(N-2)^2}{4} + \delta\right) r^{-2} v = 0$$

has a solution $v = r^{-\frac{n-2}{2}} \cos(\sqrt{\delta} \log r)$. By the oscillation theory, $u$ oscillates faster than $v$ and therefore has infinitely many roots on $[R_1, \infty)$, which is a contradiction. The proof is finished. ◻

LEMMA 11. *Assume $q > q_c > q^\star$ and $p > 1$, where $q_c$ and $q^\star$ are given in* (7a) *and* (9). *Let $u$ satisfy*

$$u'' + \frac{N-1}{r}u' - u + (\beta + \alpha r^q)u^p = 0, \ u > 0 \ \text{on } [R, \infty)$$

*for some $R \geq 0$, where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha \neq 0$. Then there exist $\delta > 0$ and $r_k \to \infty$ such that $u(r_k) < r_k^{-\frac{N-2}{2}-\delta}$ and $0 > u'(r_k) \geq -(\frac{N-2}{2} + \delta)r_k^{-\frac{N-2}{2}-\delta-1}$.*

*Proof.* First we show that $\liminf_{r\to\infty}(\beta + \alpha r^q)u^{p-1} \leq 1$. If it is false, then there are $\varepsilon > 0$ and $R_1 > R$ such that $(\beta + \alpha r^q)u^{p-1} > 1 + \varepsilon$ and $u$ oscillates fast than the solution of $v'' + \frac{N-1}{r}v' + \varepsilon v = 0$ on $[R_1, \infty)$. This is impossible since $v$ has infinitely many roots. Therefore $\liminf_{r\to\infty}(\beta + \alpha r^q)u^{p-1} \leq 1$ and there are infinitely many $\hat{r}_k \to \infty$ such that $(\beta + \alpha \hat{r}_k^q)u^{p-1}(\hat{r}_k) < 2$. That is,

$$u(\hat{r}_k) = O((\beta + \alpha \hat{r}_k^q)^{-\frac{1}{p-1}}) \leq c\, \hat{r}_k^{-\frac{N-2}{2}-\delta}$$

for some $\delta > 0$ and $c > 0$ since $q > q_c$.

If there is $R_1 > 0$ such that $u(r) \leq c\, r^{-\frac{N-2}{2}-\delta}$ for $r > R_1$, then we can find $r_k \to \infty$ such that $0 > u'(r_k) \geq -c\,(\frac{N-2}{2} + \delta)r_k^{-\frac{N-2}{2}-\delta-1}$. In this case, we also trivially have $u(r_k) \leq c\, r_k^{-\frac{N-2}{2}-\delta}$. We can remove $c$ by letting $r_1$ be big and taking a different $\delta$.

Now assume there are infinitely many $b_k \to \infty$ such that $u(b_k) = c\, b_k^{-\frac{N-2}{2}-\delta}$ and $u(r) \leq c\, r^{-\frac{N-2}{2}-\delta}$ for $b_{2k} < r < b_{2k+1}$. Since $u'(b_{2k}) \leq -(\frac{N-2}{2} + \delta)b_{2k}^{-\frac{N-2}{2}-\delta-1}$ and $u'(b_{2k+1}) \geq -(\frac{N-2}{2} + \delta)b_{2k+1}^{-\frac{N-2}{2}-\delta-1}$, there is $r_k \in [b_{2k}, b_{2k+1}]$ such that $u'(r_k) = -(\frac{N-2}{2} + \delta)r_k^{-\frac{N-2}{2}-\delta-1}$ and $u(r_k) \leq c\, r_k^{-\frac{N-2}{2}-\delta}$. Again we can remove $c$ by taking a different $\delta$. The proof is finished. ◻

LEMMA 12. *Assume $q > q_\star$ and $p^\star > p > 1$, where $q_\star$ is given in* (7a). *Let $u$ satisfy*

$$u'' + \frac{N-1}{r}u' + (\beta + \alpha r^q)u^p = 0, \ u > 0 \ \text{on } [R, \infty)$$

*for some $R \geq 0$, where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^2 + \beta^2 \neq 0$. Then there exist $\delta > 0$ and $r_k \to \infty$ such that $u(r_k) < r_k^{-\frac{N-2}{2}-\delta}$ and $0 > u'(r_k) \geq -(\frac{N-2}{2} + \delta)r_k^{-\frac{N-2}{2}-\delta-1}$.*

*Proof.* By Lemma 10, there exist $r_k \to \infty$ such that

$$(\alpha r^q + \beta)u^{p-1}(r_k) \leq \left(\frac{(N-2)^2}{4} + \delta\right) r_k^{-2}.$$

Therefore

$$u(r_k) = O((\beta r_k^2 + \alpha r_k^{q+2})^{-\frac{1}{p-1}}) \leq c r_k^{-\frac{N-2}{2}-\delta},$$

where $q > q_\star$ and $p^\star > p > 1$ are used. Now the remainder of the proof can follow the argument in the proof of Lemma 11. ◻

Now consider the problem

$$u'' + \frac{N-1}{r}u' + \alpha r^q u^p + \beta u^p = 0; \quad u(0) = 1,\ u'(0) = 0, \tag{86}$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha^2 + \beta^2 \neq 0$. The main result that we need is the following.

LEMMA 13. *Suppose that $p^\star > p > 1$ and $q > q_\star$. Then the solution to* (86) *crosses the horizontal axis.*

*Proof*. Assume that $u(r) > 0$ for $r \geq 0$. As in (16) and (17), we multiply (86) by $r^{N-1}u$ and $r^N u'$ and integrate by parts to obtain

$$-\int_0^\infty r^{N-1}u'^2 dr + \int_0^\infty r^{N-1}\left(\beta + \alpha r^q\right) u^{p+1} dr = 0 \tag{87}$$

and

$$\begin{aligned}\left(-1 + \frac{N}{2}\right)\int_0^\infty r^{N-1}u'^2 dr - \frac{N+q}{p+1}\int_0^\infty \alpha r^{N-1+q}u^{p+1}dr& \\ - \beta\frac{N}{p+1}\int_0^\infty r^{N-1}u^{p+1}dr = 0&,\end{aligned} \tag{88}$$

where the boundary terms vanish by Lemma 12. Combining (87) and (88) to eliminate the term $u'^2$, we obtain

$$\begin{aligned}&\int_0^\infty \beta r^{N-1}u^{p+1}\left[\frac{(N-2)p - (N+2)}{2(p+1)}\right]dr \\ &\qquad + \int_0^\infty \alpha r^{N-1+q}u^{p+1}\left[\frac{(N-2)p - (N+2) - 2q}{2(p+1)}\right]dr = 0.\end{aligned}$$

This is impossible since $p^\star > p > 1$ and $q > q_\star$. We have completed the proof. ◻

**Appendix C. The solution branch connecting to the positive solution for $\boldsymbol{a = 0}$.** The goal of this appendix is to rigorously prove Lemma 1. In fact we will consider a slightly more general problem. For convenience, we let $h(r; a) = 1 + ar^q$ and we consider the problem

$$u_{rr} + \frac{N-1}{r}u_r - u + h(r; a)|u|^p = 0, u > 0, u_r(0) = 0, \lim_{r\to\infty} u(r) = 0. \tag{89}$$

When $a = 0$, by [17], the equation has a unique solution $U(r)$. In this section, we restrict ourselves to radially symmetric functions and are concerned with the local existence of the solution branch of (89) for $a > 0$ which connects to $U(r)$. Let $s_0 = U(0)$. The proof of Lemma 1 relies on Lemmas 14 and 15, which we now prove.

LEMMA 14. *Let $u$ be a solution of* (89)*. Assume $h(r; a)[u(r)]^{p-1} \leq 1 - \tau$ for $r \geq R$, where $R > 0$ and $1 > \tau > 0$. Then*

$$u(r) \leq \left(\frac{1-\tau}{h(R; a)}\right)^{\frac{1}{p-1}} e^{\alpha(R-r)} \text{ for } r \geq R,$$

*where $\alpha = \sqrt{\tau}$. Moreover, for any $\alpha_1 < 1$, there is $c > 0$ such that $u(r) \leq ce^{-\alpha_1 r}$ on* $[0, \infty)$.

*Proof.* Let

$$w_1 = \left(\frac{1-\tau}{h(R;a)}\right)^{\frac{1}{p-1}} e^{\alpha(R-r)} \text{ and } w = u - w_1.$$

Then on $[R, \infty)$, $w_1$ satisfies

$$\Delta w_1 - w_1 + hu^{p-1} w_1 \leq (w_1)_{rr} - \tau w_1 = 0$$

and $w$ satisfies

$$\Delta w - w + hu^{p-1} w \geq 0.$$

Note that $w(R) \leq 0$ and $\lim_{r\to\infty} w(r) = 0$. By the maximum principle, $w \leq 0$ on $[R, \infty)$.

Since $u$ decays exponentially, $hu^{p-1} \to 0$ as $r \to \infty$. We can apply the argument above to conclude that for any $\alpha_1 < 1$, there is $c > 0$ such that $u(r) \leq ce^{-\alpha_1 r}$ on $[0, \infty)$. ◻

LEMMA 15. *Let $\bar{a} \geq 0$ and $\bar{u}$ be a solution of* (89) *with $a = \bar{a}$. Assume $\bar{u}$ has exponential decay and the solution $Z$ of the linearized equation*

$$L_{\bar{u},\bar{a}} Z := Z_{rr} + \frac{N-1}{r} Z_r + (ph(r;\bar{a})\bar{u}^{p-1} - 1)Z = 0, Z(0) = 1, Z_r(0) = 0, \tag{90}$$

*satisfies* $\lim_{r\to\infty} |Z(r)| = \infty$. *Then there exists $\delta > 0$ such that* (89) *has a positive exponential decay solution $u(r;a)$ for $|a - \bar{a}| < \delta$. Moreover, $u(r;a)$ is $C^1$ in the variable $a$.*

*Proof.* Let $u(r;a) = \bar{u}(r;\bar{a}) + v(r;a)$. Then $u(r;a)$ satisfies (89) iff $v$ is a solution of

$$\begin{aligned} L_{\bar{u},\bar{a}} v &= g(r, v(r), a) \\ &=: -h(r;\bar{a})[(\bar{u}+v)^p - \bar{u}^p - p\bar{u}^{p-1} v] + [h(r;\bar{a}) - h(r;a)](\bar{u}+v)^p, \\ v_r(0) &= 0. \end{aligned} \tag{91}$$

Let $Z$ be the solution of (90) and let $Z_1$ satisfy $L_{\bar{u},\bar{a}} Z_1 = 0$ such that the Wronskian $W(Z_1, Z)$ of $Z_1$ and $Z$ satisfies $W(Z_1, Z) = r^{-N+1}$. Then as $r \to 0$, we have $|Z_1(r)| = O(r^{-N+2})$ for $N \geq 3$; $|Z_1(r)| = O(-\log r)$ for $N = 2$; $|Z_1(r)| = O(1)$ for $N = 1$. Since $\bar{u}$ decays exponentially, $L_{\bar{u},\bar{a}} \sim \Delta - 1$ for large $r$. By ODE theory, $Z(r) = O(e^{(1+\epsilon_1)r})$ and $Z_1(r) = O(e^{-(1-\epsilon_1)r})$ for any small $\epsilon_1 > 0$. Also, ODE theory implies $\bar{u}(r) = O(e^{-(1-\epsilon_1)r})$. The method of variation of parameters yields the formula for $v$ in terms of $g$,

$$v(r) = -Z_1(r) \int_0^r \frac{Z(\eta) g(\eta, v(\eta), a)}{W(Z_1, Z)(\eta)}\, d\eta + Z(r) \int_0^r \frac{Z_1(\eta) g(\eta, v(\eta), a)}{W(Z_1, Z)(\eta)}\, d\eta + \beta Z + \gamma Z_1. \tag{92}$$

To seek an exponential decay solution, we have to eliminate the term $Z$ as $r \to \infty$. Therefore we take

$$\beta = -\int_0^\infty \eta^{N-1} Z_1(\eta) g(\eta, V(\eta), a)\, d\eta. \tag{93}$$

To let $v(0)$ remain bounded and $v_r(0) = 0$, we take $\gamma = 0$. That is, $v$ should satisfy

$$v(r) = H(v, a) =: -Z_1(r) \int_0^r \eta^{N-1} Z(\eta) g\, d\eta - Z(r) \int_r^\infty \eta^{N-1} Z_1(\eta) g\, d\eta. \tag{94}$$

To prove that (94) has a solution, let $0 < \alpha < 1$ and consider the weighted norm

$$|v|_\alpha = \sup_{r \geq 0} |v(r)e^{\alpha r}| \tag{95}$$

and the space

$$L^\infty_\alpha = \{v \text{ is continuous on } [0, \infty) : |v|_\alpha < \infty\}. \tag{96}$$

Let $\alpha = 1 - \epsilon_1$ and $\epsilon_1 > 0$ be small enough such that $p\alpha > 1 + 2\epsilon_1$. To solve (94), we show that $H(v, a)$ is a contraction mapping on a ball in $L^\infty_\alpha$. Let $v_1, v_2 \in L^\infty_\alpha$ with $|v_1|_\alpha, |v_2|_\alpha \leq \delta$, where $\delta$ is to be chosen later. Using the facts $Z(r) = O(e^{(1+\epsilon_1)r})$, $Z_1(r) = O(e^{-(1-\epsilon_1)r})$, $\bar{u}(r) = O(e^{-(1-\epsilon_1)r})$ as $r \to \infty$, and $|v_1(r)|, |v_2(r)| \leq \delta e^{-\alpha r}$, we have the following estimates:

$$\begin{aligned} &|g(r, v_1(r), a) - g(r, v_2(r), a)| \\ &\quad \leq c\, h(r; \bar{a})(\bar{u}(r) + |v_1(r)| + |v_2(r)|)^{p-2}(|v_1(r)| + |v_2(r)|)|v_1(r) - v_2(r)| \\ &\qquad + c\, r^q |a - \bar{a}|(\bar{u}(r) + |v_1(r)| + |v_2(r)|)^{p-1}|v_1(r) - v_2(r)| \\ &\quad \leq \hat{c}\, h(r; \bar{a}) e^{-p\alpha r} \delta^{\min\{p-1,1\}} |v_1 - v_2|_\alpha + \hat{c}\, r^q e^{-p\alpha r} |a - \bar{a}||v_1 - v_2|_\alpha \end{aligned} \tag{97}$$

with some constants $c$ and $\hat{c}$, which can be verified by considering the case $\bar{u}(r) > 2(|v_1(r)| + |v_2(r)|)$ and the case $\bar{u}(r) \leq 2(|v_1(r)| + |v_2(r)|)$ separately.

Denote the first term and second term in $H(v, a)$ by $F_1(v, a)$ and $F_2(v, a)$, respectively. From the above estimate and the fact that $p\alpha > 1 + 2\epsilon_1$, we have

$$\begin{aligned} &|F_1(v_1, a) - F_1(v_2, a)| \\ &\quad \leq c_1 |Z_1(r)| \delta^{\min\{p-1,1\}} |v_1 - v_2|_\alpha \int_0^r \eta^{N-1}(1 + \eta^q)|Z(\eta)| e^{-p\alpha\eta}\, d\eta \\ &\qquad + c_2 |a - \bar{a}||Z_1(r)||v_1 - v_2|_\alpha \int_0^r \eta^{N-1+q} |Z(\eta)| e^{-p\alpha\eta}\, d\eta \\ &\quad \leq c_3 e^{-\alpha r}(\delta^{\min\{p-1,1\}} + |a - \bar{a}|)|v_1 - v_2|_\alpha r^{N-1}(1 + r^q) e^{(-p\alpha + 1 + \epsilon_1)r} \\ &\quad \leq c_4 r^{N-1}(1 + r^q) e^{-(\alpha + \epsilon_1)r}(\delta^{\min\{p-1,1\}} + |a - \bar{a}|)|v_1 - v_2|_\alpha \end{aligned} \tag{98}$$

and

$$\begin{aligned} &|F_2(v_1, a) - F_2(v_2, a)| \\ &\quad \leq c_5 |Z(r)| \delta^{\min\{p-1,1\}} |v_1 - v_2|_\alpha \int_r^\infty \eta^{N-1}(1 + \eta^q)|Z_1(\eta)| e^{-p\alpha\eta}\, d\eta \\ &\qquad + c_6 |a - \bar{a}||Z(r)||v_1 - v_2|_\alpha \int_r^\infty \eta^{N-1+q} |Z_1(\eta)| e^{-p\alpha\eta}\, d\eta \\ &\quad \leq c_7 e^{(1+\epsilon_1)r}(\delta^{\min\{p-1,1\}} + |a - \bar{a}|)|v_1 - v_2|_\alpha (1 + r^{N-1+q}) e^{-(p+1)\alpha r} \\ &\quad \leq c_8 (1 + r^{N-1+q}) e^{-(\alpha + \epsilon_1)r}(\delta^{\min\{p-1,1\}} + |a - \bar{a}|)|v_1 - v_2|_\alpha. \end{aligned} \tag{99}$$

Therefore

$$\begin{aligned} |H(v_1, a) - H(v_2, a)|_\alpha &\leq |F_1(v_1, a) - F_1(v_2, a)|_\alpha + |F_2(v_1, a) - F_2(v_2, a)|_\alpha \\ &\leq c_9(\delta^{\min\{p-1,1\}} + |a - \bar{a}|)|v_1 - v_2|_\alpha \\ &\leq \frac{1}{2}|v_1 - v_2|_\alpha \end{aligned} \tag{100}$$

if $\delta$ is small, $|a-\bar{a}|<\delta$, and $|v_1|_\alpha, |v_2|_\alpha \le \delta$. For $v=0$, we have

$$|H(0,a)| \le |a-\bar{a}||Z_1(r)| \int_0^r \eta^{N-1+q}|Z(\eta)|\bar{u}^p\, d\eta + |a-\bar{a}||Z(r)| \int_r^\infty \eta^{N-1+q}|Z_1(\eta)|\bar{u}^p\, d\eta \le c_{10}|a-\bar{a}|(1+r^{N-1+q})e^{-(\alpha+\epsilon_1)r}. \tag{101}$$

This implies

$$|H(0,a)|_\alpha \le c_{11}|a-\bar{a}| < \frac{1}{4}\delta \tag{102}$$

if we further assume $|a-\bar{a}| \le \delta_0 = \dfrac{\delta}{4c_{11}+1}$. For $|v|_\alpha \le \delta$, we have

$$|H(v,a)|_\alpha \le |H(v,a)-H(0,a)|_\alpha + |H(0,a)|_\alpha < \frac{1}{2}|v-0|_\alpha + \frac{1}{4}\delta \le \frac{3}{4}\delta. \tag{103}$$

The above estimates show that $H(v,a)$ is a contraction mapping defined on the ball $\{v \in L^\infty_\alpha : |v|_\alpha \le \delta\}$. The fixed point theorem then implies (94) has a solution $v(r;a)$ and (89) has an exponential decay solution $u(r:a) = \bar{u}(r;\bar{a}) + v(r;a)$ for $|a-\bar{a}| < \delta_0$ and $a \ge 0$.

The positivity of $u(r;a)$ is shown in Lemma 4. The $C^1$ property of $u(r;a)$ in $a$ follows from a standard argument in the implicit function theory for a Banach space. ∎

*Proof of Lemma* 1. For a given $(a,s)$, let $v(r;a,s)$ denote the solution to

$$v_{rr} + \frac{N-1}{r}v_r - v + |v|^p\, h(r) = 0, \quad v'(0)=0, \quad v(0)=s, \quad r>0. \tag{104}$$

Let $U(r)$ be the unique solution of (89) for $a=0$ and let $s_0 = U(0)$. It is known that for $a=0$, the solution $Z(r)$ of the corresponding linearized equation with $Z(0)=1$ and $Z_r(0)=0$ satisfies the property $\lim_{r\to\infty} Z(r) = -\infty$. Therefore by applying Lemma 15 to the case $\bar{u}=U$ and $\bar{a}=0$, Lemma 1 is proved. ∎

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